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A superstatistical model for anomalous heat conduction and diffusion

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ABSTRACT

We propose a superstatistical model for anomalous heat conduction and diffusion, which is formulated by the thermal conductivity distribution, overall temperature and heat flux distributions. Our model obeys Fourier's law and the continuity equation at the individual level. The evolution of the thermal conductivity distribution is described by an advection-diffusion equation. We show that the superstatistical model predict anomalous behaviors including the time-dependent effective thermal conductivity and slow long-time asymptotics. The time-dependence of the effective thermal conductivity is determined by the mean square displacement (MSD), which coincides with existing investigations. The superstatistical structure can also be extended into other non-Fourier models including the Cattaneo and fractional-order heat conduction models.

1. Introduction

Anomalous diffusion has been intensively discussed in the past decades. It has been observed in a wide class of systems, such as transport of fluid in porous media [1–3], diffusion in plasmas [4], subregion laser cooling [5], heart-beat histograms [6], and turbulent flows [7]. One common class of anomalous diffusion is characterized by the temporal growth of the mean square displacement (MSD), namely, $\langle|\Delta\mathbf{x}|^2\rangle \propto t^\alpha$. The range of α is often classified into five subranges: hyperdiffusion, $\alpha > 2$; ballistic motion, $\alpha = 2$; superdiffusion, $\alpha > 1$; Brownian diffusion, $\alpha = 1$; and subdiffusion, $0 < \alpha < 1$. There are many approaches to model the non-Brownian MSD. Fractional Brownian motion (FBM) [8] is a typical one, which satisfies $\langle|\Delta\mathbf{x}|^2\rangle \sim t^{2H}$ with $0 < H < 1$ denoting the Hurst exponent. Another well-known model is the continuous-time random walk (CTRW) [9], where the length of random jumps is given by a probability density with finite second moment. The Fokker-Planck equation (FPE) for FBM possesses a time-dependent effective diffusivity, while the CTRW usually corresponds to the fractional Fokker-Planck equation (FFPE). Although both of them can give rise to a non-Brownian MSD, their physical meanings are fundamentally different. Fractional-order derivatives in the FFPE reflect memory or nonlocal behaviors, which do not exist in FBM. Based on the so-called p variations, Magdziarz et al. [10] proposed a test to distinguish the two models in subdiffusive dynamics, and they have concluded that FBM is in better correspondence with the previous data than the CTRW. Besides the two classical approaches, nonlinear generalizations of the FPE also form a paradigmatic class [11], which can be associated with the non-extensive Tsallis statistical mechanics.

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The non-Brownian MSD has been connected to anomalous heat conduction as well. General heat conduction is described by Fourier's law,

$$\mathbf{q}(\mathbf{x}, t) = -\lambda \nabla T(\mathbf{x}, t), \tag{1}$$

wherein $\mathbf{q}(\mathbf{x}, t)$ is the local heat flux density, λ is the thermal conductivity and $T(\mathbf{x}, t)$ is the local temperature. In Eq. (1), λ is an intrinsic material property, which can be temperature-dependent or temperature-independent. In anomalous heat conduction, there exists no well-defined λ due to the low-dimensional effects [12–16], and the effective thermal conductivity λ_{eff} is therefore introduced instead. The effective thermal conductivity λ_{eff} is commonly formulated in an one-dimensional (1D) heat conduction problem with a constant temperature difference ΔT, namely,

$$q_x = -\lambda_{eff}\frac{\Delta T}{L}, \tag{2}$$

wherein q_x denotes the 1D heat flux and L is the length along ΔT. Obviously, λ_{eff} reduces to λ in Fourier heat conduction, while anomalous heat conduction gives rise to length-dependent or time-dependent λ_{eff} [12–16]. The length-dependence or time-dependence can be determined by the non-Brownian exponent α, i.e., $\lambda_{eff} \propto L^{2-2/\alpha}$ [12] and $\lambda_{eff} \propto t^{\alpha-1}$ [13]. Therefore, descriptions of anomalous diffusion such as the fractional-order derivative have been applied to modeling anomalous heat conduction as well [17–19]. Using the Riemann-Liouville (RL) operator on the right-hand side ${}_0^{RL}D_t^{1-\beta}$ $(0 < \beta < 1)$, Eq. (1) can be generalized into

$$\mathbf{q}(\mathbf{x}, t) = \tau_\beta^{1-\beta}\,{}_0^{RL}D_t^{1-\beta}[\lambda \nabla T(\mathbf{x}, t)] = -\frac{\tau_\beta^{1-\beta}}{\Gamma(\beta)}\frac{\partial}{\partial t}\int_0^t (t-\xi)^{\beta-1}\lambda \nabla T(\mathbf{x}, \xi)d\xi, \tag{3}$$

where τ_β is a positive constant. For a time-independent temperature distribution, this model leads to

$$\mathbf{q}(t) = -\frac{\lambda}{\Gamma(\beta)}\left(\frac{t}{\tau_\beta}\right)^{\beta-1}\nabla T(\mathbf{x}), \tag{4}$$

and the effective thermal conductivity defined by Eq. (2) reads $\lambda_{eff} \propto t^{\beta-1}$. Eq. (3) is the simplest fractional heat conduction model, which has been widely discussed and further generalized [18,19].

Recently, a new subclass of anomalous diffusion has attracted increasing interest, anomalous yet Brownian diffusion [20–27], which fulfills the linear growth of the MSD yet coexists with a non-Gaussian distribution. It implies that $\alpha = 1$ is not enough to describe standard diffusion, which also requires the Gaussian probability density function (PDF) at least,

$$P(\mathbf{x}, t|D) = (4\pi Dt)^{-\frac{N}{2}}\exp\left(-\frac{|\mathbf{x}|^2}{4Dt}\right), \tag{5}$$

wherein N is the spatial dimension and D stands for the corresponding diffusivity. The Brownian yet non-Gaussian behavior has been observed in various systems such as networks [20], biological cells [21], price dynamics [22], dense granular flows [23] and super-cooled liquids [24]. How to model this behavior remains an intriguing issue, and one remarkable approach is in the framework of superstatistics. The superstatistical formulism introduces a diffusivity distribution, $\rho_D = \rho_D(D)$, and the overall distribution function $P_o(\mathbf{x},t)$ is expressed by the averaging procedure of $P(\mathbf{x}, t|D)$ over D, namely,

$$P_o(\mathbf{x}, t) = \int_0^{+\infty} P(\mathbf{x}, t|D)\rho_D(D)dD. \tag{6}$$

Through appropriate choices of $\rho_D(D)$, non-Gaussian shapes [27] such as the Laplace and power-law distributions can emerge from Eq. (6). It is not difficult to demonstrate that $P_o(\mathbf{x},t)$ possesses the same asymptotics of the MSD as the standard distribution $P(\mathbf{x}, t|D)$:

$$\begin{aligned}\int \left(\Delta\mathbf{x}^2\right)P_o(\mathbf{x}, t)d\mathbf{x}^N &= \int \left(\Delta\mathbf{x}^2\right)\left[\int_0^{+\infty} P(\mathbf{x}, t|D)\rho_D(D)dD\right]d\mathbf{x}^N \\ &= \int_0^{+\infty}\left[\int \left(\Delta\mathbf{x}^2\right)P(\mathbf{x}, t|D)d\mathbf{x}^N\right]\rho_D(D)dD \propto t.\end{aligned} \tag{7}$$

In non-Fourier heat conduction [28–31], the superstatistical framework has not been discussed but there exists similar formalism. For instance, the phonon Boltzmann transport equation (BTE) with the relaxation time approximation [32] predicts the effective thermal conductivity as the sum of the contributions from each phonon mode q:

$$\lambda_{eff} = \sum_q^{N_q}\lambda_q = \sum_q^{N_q}|v_q|^2 c_q \tau_q, \tag{8}$$

where N_q is the total number of phonon modes, λ_q is the contribution of mode q, v_q is the group velocity of mode q, c_q is the specific heat of mode q and τ_q is the relaxation time of mode q. In the sprit of superstatistics, Eq. (8) arises from distributed λ, and the thermal conductivity distribution ρ_λ is given by

$$\rho_\lambda = \frac{1}{N_q}\sum_{q}^{N_q}\delta(\lambda - N_q\lambda_q). \tag{9}$$

The continuous case can be found in renormalized phonons and effective phonon theory [33], which introduces a weight factor $\rho_q = \rho_q(q)$ in the conventional Deybe formula,

$$\lambda_{eff} = \frac{1}{2\pi}\int_0^{2\pi}|v_q|^2 c_q\tau_q\rho_q(q)dq \tag{10}$$

If there exists a differentiable inverse function $q = q(\lambda)$, one can write the thermal conductivity distribution as

$$\rho_\lambda = \rho_q\frac{dq}{d\lambda}. \tag{11}$$

Accordingly, the superstatistical models can be applied to heat transport wherein the effective thermal conductivity consists of contributions from different parts. It should be mentioned that anomalous heat conduction is conceptually different from anomalous diffusion [12]. The aim of discussing anomalous diffusion is to introduce the background of the superstatistical approach.

The main aim of this work is to introduce a superstatistical approach to modeling heat conduction and discuss its implications and possible applications. The superstatistical framework consists of the thermal conductivity distribution, overall temperature and heat flux distributions. The evolution of the thermal conductivity distribution is described by an advection-diffusion equation. We show that the superstatistical heat conduction model can predict anomalous λ_{eff} related to the non-Brownian MSD, which coincides with several previous investigations. The entropy production is also discussed and the superstatistical model is extended into other non-Fourier models.

2. Superstatistical model of heat conduction

2.1. Superstatistical formalism

We first introduce the overall temperature distribution $T_o(\mathbf{x},t)$ and overall heat flux $\mathbf{q}_o(\mathbf{x},t)$, which are written as

$$T_o(\mathbf{x},t) = \int_0^{+\infty} T(\mathbf{x},t|\lambda)\rho_\lambda d\lambda, \tag{12}$$

$$\mathbf{q}_o(\mathbf{x},t) = \int_0^{+\infty} \mathbf{q}(\mathbf{x},t|\lambda)\rho_\lambda d\lambda. \tag{13}$$

ρ_λ must satisfy the normalization condition $\int_0^{+\infty}\rho_\lambda d\lambda = 1$ and non-negativity, $\inf(\rho_\lambda) \geq 0$. The average of λ is given by $\langle\lambda\rangle = \int_0^{+\infty}\lambda\rho_\lambda d\lambda$, and when $\rho_\lambda = \delta(\lambda - \langle\lambda\rangle)$, it reduces to Fourier's law of heat conduction. We propose that $T(\mathbf{x}, t|\lambda)$ and $\mathbf{q}(\mathbf{x}, t|\lambda)$ obeys Fourier heat conduction at the individual level,

$$\mathbf{q}(\mathbf{x},t|\lambda) = -\lambda\nabla T(\mathbf{x},t|\lambda), \tag{14}$$

and the continuity equation (pure heat conduction)

$$c\frac{\partial T(\mathbf{x},t|\lambda)}{\partial t} = -\nabla\cdot\mathbf{q}(\mathbf{x},t|\lambda), \tag{15}$$

wherein c stands for the specific heat capacity per volume. Combining Eq. (14) with Eq. (15) yields

$$c\frac{\partial T(\mathbf{x},t|\lambda)}{\partial t} = \nabla\cdot[\lambda\nabla T(\mathbf{x},t|\lambda)], \tag{16}$$

Upon multiplying Eq. (15) by ρ_λ and integrating it over λ, one can find that $T_o(\mathbf{x},t)$ and $\mathbf{q}_o(\mathbf{x},t)$ satisfy the continuity equation:

$$\begin{aligned} c\frac{\partial T_o(\mathbf{x},t)}{\partial t} &= \int_0^{+\infty} c\frac{\partial T(\mathbf{x},t|\lambda)}{\partial t}\rho_\lambda d\lambda \\ &= \int_0^{+\infty} -\nabla\cdot\mathbf{q}(\mathbf{x},t|D)\rho_\lambda d\lambda \\ &= -\nabla\cdot\int_0^{+\infty}\mathbf{q}(\mathbf{x},t|D)\rho_\lambda d\lambda = -\nabla\cdot\mathbf{q}_o(\mathbf{x},t). \end{aligned} \tag{17}$$

In contrast with the previous investigations, our model does not focus on the distribution of the thermal diffusivity, which is expressed as $D_{th} = \lambda/c$. That is because temperature-dependent material properties, i.e., $\lambda = \lambda(T)$ and $c = c(T)$, are very common in Fourier heat conduction. Then, the heat conduction equation deviates from the standard diffusive form $\partial T/\partial t = D_{th}\nabla^2 T$, namely,

$$\frac{\partial T}{\partial t} = D_{th}(T)\left[\nabla^2 T + \frac{d\ln c(T)}{dT}|\nabla T|^2\right] + \frac{dD_{th}(T)}{dT}|\nabla T|^2. \tag{18}$$

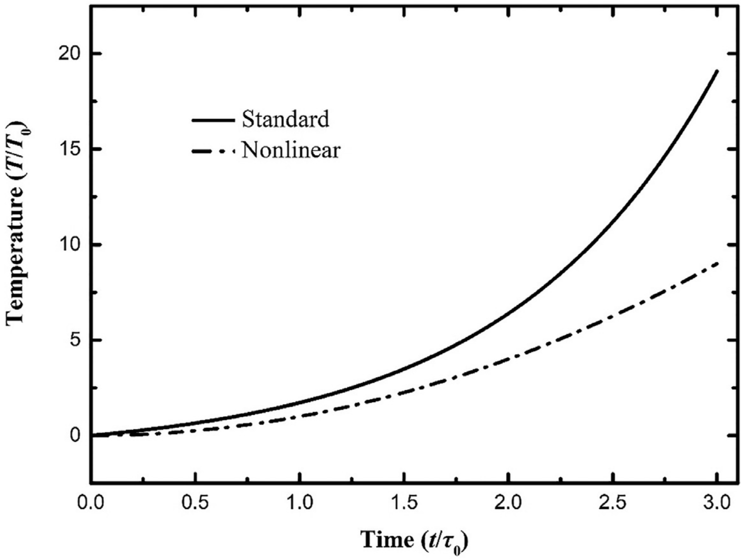

Fig. 1. Traveling wave solution.

Owing to the nonlinearity, the features of heat conduction can be fundamentally different from linear heat diffusion. Examples include fast and superfast diffusion [34]. There are different choices of temperature-dependent material properties. The kinetic situation with $\kappa(T) \propto \sqrt{T}$ is taken as an example. In Fig. 1, we show the difference between the standard and nonlinear cases by their traveling wave solutions at $x=0$. The traveling wave solutions take the form $\frac{T}{T_0} = \varsigma(\frac{x}{l_0} + v\frac{t}{\tau_0})$, where $\varsigma(\frac{x}{l_0} + v\frac{t}{\tau_0})$ is a dimensionless function, T_0, l_0 and τ_0 are constants. There is an obvious deviation between the standard and nonlinear solutions in the long-time limit. More examples can be found in Ref. [35]. Hence the usual physical understanding of the thermal diffusivity (the coefficient of diffusive transport) no longer holds, whereas the meaning of the thermal conductivity remains unchanged. Furthermore, the thermal diffusivity is the ratio of the thermal conductivity and specific heat capacity per volume. However the thermal conductivity corresponds to the law of heat conduction, while the specific heat capacity per volume relates to energy conservation. If we introduce a superstatistical formulation involving the thermal diffusivity, it will be ambiguous whether the superstatistical structure emerges from a superstatistical model based on the constitutive equation relating $T(\mathbf{x}, t)$ and $\mathbf{q}(\mathbf{x}, t)$ or the continuity equation.

Because $T(\mathbf{x}, t|\lambda)$ and $\mathbf{q}(\mathbf{x}, t|\lambda)$ depend on distributed λ, we cannot establish an explicit constitutive relation between $T_o(\mathbf{x},t)$ and $\mathbf{q}_o(\mathbf{x},t)$ in generic cases. For steady-state heat conduction with temperature-independent λ, Eq. (16) reduces to $\nabla^2 T(\mathbf{x},t|\lambda)=0$, whose solution only depends on the boundary conditions. If the boundary conditions are independent of λ, the solution will be independent of λ. Then, we acquire

$$\begin{aligned}\mathbf{q}_o(\mathbf{x}, t) &= \int_0^{+\infty} \mathbf{q}(\mathbf{x}, t|\lambda)\rho_\lambda d\lambda \\ &= -\int_0^{+\infty} \lambda \nabla T(\mathbf{x}, t|\lambda)\rho_\lambda d\lambda \\ &= -\left(\int_0^{+\infty} \lambda\rho_\lambda d\lambda\right)\nabla T_o(\mathbf{x}) = -\langle\lambda\rangle \nabla T_o(\mathbf{x}).\end{aligned} \tag{19}$$

For Eq. (2), This special case leads to $\lambda_{eff} = \langle\lambda\rangle$, where $\langle\lambda\rangle = \int_0^{+\infty} \lambda\rho_\lambda d\lambda$ is the average of the thermal conductivity.

2.2. Diffusing thermal conductivity

In this section, we focus on the temporal evolution of the thermal conductivity distribution, $\rho_\lambda = \rho_\lambda(\lambda,t)$. Chubynsky and Slater [26] have proposed a simple and generic description for anomalous yet Brownian diffusion, which is called the diffusing diffusivity model. The diffusing diffusivity model assumes that a diffusivity of the tracer particle evolves in time like a Brownian particle in an external force field. It requires that the diffusivity varies little from step to step enabling us to switch to a continuous time. Finally, the evolution of $\rho_\lambda(\lambda,t)$ will satisfy the advection-diffusion equation as follows

$$\frac{\partial \rho_D(D,t)}{\partial t} = \frac{\partial^2[K_D(D)\rho_D(D,t)]}{\partial D^2} + \frac{\partial[s_D(D)\rho_D(D,t)]}{\partial D}, \tag{20}$$

where $K_D(D)$ is the diffusivity of the thermal diffusivity and $-s_D(D)$ is the bias of diffusion of the thermal diffusivity. Similarly, we assume the same form as Eq. (20) for $\rho_\lambda(\lambda,t)$:

$$\frac{\partial \rho_\lambda(\lambda,t)}{\partial t} = \frac{\partial^2[K_\lambda(\lambda)\rho_\lambda(\lambda,t)]}{\partial \lambda^2} + \frac{\partial[s_\lambda(\lambda)\rho_\lambda(\lambda,t)]}{\partial \lambda}, \tag{21}$$

where $K_\lambda(\lambda)$ is the diffusivity of the thermal conductivity and $-s_\lambda(\lambda)$ is the bias of diffusion of the thermal conductivity. The diffusing diffusivity model enables the diffusivity to have a finite upper bound, $D \in [0, D_{\max}]$, while in this work, we set $\lambda_{\max} \to +\infty$ in order to simplify the discussion. For the temperature-independent λ and time-independent temperature distribution, one can acquire

$$\mathbf{q}_o(\mathbf{x},t) = -\left(\int_0^{+\infty} \lambda\rho_\lambda(\lambda,t)d\lambda\right)\nabla T_o(\mathbf{x}) = -\langle\lambda(t)\rangle\nabla T_o(\mathbf{x}). \tag{22}$$

Eq. (22) describes a possible anomalous behavior which allows a time-dependent heat flux in a time-independent temperature field. The simplest case of Eq. (22) is that of constant $K_\lambda(\lambda)$ and $s_\lambda(\lambda)$, which leads to an exponential stationary diffusivity distribution,

$$\rho_\lambda(\lambda) = \frac{1}{\langle\lambda\rangle}\exp\left(-\frac{\lambda}{\langle\lambda\rangle}\right),\ \langle\lambda\rangle = \frac{K_\lambda}{s_\lambda}. \tag{23}$$

Another case is the porous media equation [26], wherein $s_\lambda(\lambda)=0$ and $K_\lambda(\lambda)=k\lambda^m$ ($k > 0$ and $m \neq 2$ are constants). The corresponding quasi-stationary solution of Eq. (21) takes the following form

$$\rho_\lambda(\lambda,t) = t^n f(\lambda t^n),\ n = \frac{1}{m-2}. \tag{24}$$

The self-similar function $f(\chi)$ is given by the following ordinary differential equation (ODE)

$$\frac{k\chi^m}{n}\frac{d^2 f(\chi)}{d\chi^2} + a_1\frac{df(\chi)}{d\chi} + a_0 f(\chi) = 0 \tag{25}$$

with

$$a_1 = \left[\frac{2k(m-1)\chi^{m-1}}{n} - \chi\right],\ a_0 = \left[\frac{km(m-1)\chi^{m-2}}{n} - 1\right]. \tag{26}$$

The solution-determining conditions of Eq. (25) are given by the normalization condition

$$\int_0^{+\infty} \rho_\lambda(\lambda,t)d\lambda = \int_0^{+\infty} f(\chi)d\chi = 1, \tag{27}$$

and the mean condition

$$\langle\lambda(t)\rangle = \int_0^{+\infty} \lambda\rho_\lambda(\lambda,t)d\lambda = t^{-n}\int_0^{+\infty} \chi f(\chi)d\chi. \tag{28}$$

The anomalous behaviors induced by $\rho_\lambda(\lambda,t)$ will be discussed in Section 3.

2.3. Connection to existing theory

In existing investigations, the superstatistical formalism is for the PDF, $P(\mathbf{x}, t|D)$, while in our model, the PDF is replaced by the temperature distribution. In the following, we will discuss the connection between the superstatistical frameworks for the PDF and temperature distribution. Our discussion relies on classical irreversible thermodynamics (CIT) [36] and Boltzmann-Gibbs (BG) statistical mechanics [37]. Consider the entropy density $s(\mathbf{x}, t|\lambda)$ and entropy flux $\mathbf{J}_S(\mathbf{x},t|\lambda)$ in Fourier heat conduction at the individual level. In the framework of CIT, $s(\mathbf{x}, t|\lambda)$ and $\mathbf{J}_S(\mathbf{x},t|\lambda)$ can be estimated by local thermodynamic quantities:

$$s(\mathbf{x}, t|\lambda) \cong \int^{T(\mathbf{x},t|\lambda)} c\frac{dT}{T}, \tag{29}$$

$$\mathbf{J}_S(\mathbf{x}, t|\lambda) \cong \frac{\mathbf{q}(\mathbf{x}, t|\lambda)}{T(\mathbf{x}, t|\lambda)}. \tag{30}$$

Substituting Eq. (14) into (30) yields

$$\begin{aligned}\mathbf{J}_S(\mathbf{x}, t|\lambda) &= \frac{\mathbf{q}(\mathbf{x}, t|\lambda)}{T(\mathbf{x}, t|\lambda)} = -\frac{\lambda\nabla T(\mathbf{x}, t|\lambda)}{T(\mathbf{x}, t|\lambda)} \\ &= -\frac{\lambda}{c}\frac{c\nabla T(\mathbf{x}, t|\lambda)}{T(\mathbf{x}, t|\lambda)} = -\frac{\lambda}{c}\nabla\left(\int^{T(\mathbf{x},t|\lambda)} c\frac{dT}{T}\right) = -D_{th}\nabla s(\mathbf{x}, t|\lambda).\end{aligned} \tag{31}$$

Thus, superstatistical Fourier heat conduction corresponds to superstatistical entropy transport. Note that the deduction in Eq. (31) is still valid for temperature-dependent λ and c. Although the usual understanding of $D_{th} = \lambda/c$ may not hold for temperature-dependent material properties, it can always be considered as the transport coefficient of entropy transport.

We now focus on the evolution of the PDF in the absence of an external force field, which is generally modeled in terms of the following force-free FPE

$$\frac{\partial P(\mathbf{x}, t|D)}{\partial t} = \nabla \cdot [D\nabla P(\mathbf{x}, t|D)]. \tag{32}$$

The Gaussian PDF is a typical solution of Eq. (32), and in heat conduction, the PDF can be defined as the normalization of the energy correlation function $C_{ee}(\mathbf{x},t|D)$ [13], namely,

$$P(\mathbf{x}, t|D) = \left[\int C_{ee}(\mathbf{x}, t = 0|D) d^N\mathbf{x}\right]^{-1} C_{ee}(\mathbf{x}, t|D). \tag{33}$$

In Dhar's review [14], the force-free FPE is used to derive the Green-Kubo formula, while we here consider its superstatistical generalization. Eq. (32) arises from the following constitutive equation

$$\mathbf{J} = \mathbf{J}(\mathbf{x}, t|D) = -D\nabla P(\mathbf{x}, t|D) \tag{34}$$

with $\mathbf{J}(\mathbf{x}, t|D)$ denoting the probability current. In BG statistical mechanics [37], the entropy of the system is given by

$$S(t|D) = \int -k_B P(\mathbf{x}, t|D) \ln P(\mathbf{x}, t|D) d^N\mathbf{x}, \tag{35}$$

with k_B is the Boltzmann constant. Thereby, the entropy density reads

$$s(\mathbf{x}, t|D) = -k_B P(\mathbf{x}, t|D) \ln P(\mathbf{x}, t|D), \tag{36}$$

whose time derivative is written as

$$\begin{aligned}\frac{\partial s}{\partial t} &= -k_B(\ln P + 1)\frac{\partial P}{\partial t} = k_B(\ln P + 1)\nabla \cdot \mathbf{J} \\ &= \nabla \cdot [k_B\mathbf{J}(\ln P + 1)] - k_B\mathbf{J} \cdot \nabla(\ln P).\end{aligned} \tag{37}$$

In general, the total entropy production rate of the system is given by [38]

$$\Phi = k_B \int \frac{\mathbf{J} \cdot \mathbf{J}}{DP} d^N\mathbf{x} = -k_B \int \mathbf{J} \cdot \nabla(\ln P) d^N\mathbf{x}, \tag{38}$$

and hence the local entropy production rate σ takes the form $\sigma = -k_B\mathbf{J} \cdot \nabla(\ln P)$. Upon combining σ with the entropy balance equation

$$\frac{\partial s}{\partial t} = -\nabla \cdot \mathbf{J}_S + \sigma, \tag{39}$$

we obtain

$$\mathbf{J}_S = -k_B\mathbf{J}(\ln P + 1) = Dk_B(\ln P + 1)\nabla P = -D\nabla s, \tag{40}$$

which agrees with Eq. (31) when $D = D_{th}$. Therefore, the superstatistical heat conduction can be connected to the superstatistical theory for the PDF through Dhar's model of energy fluctuations.

3. Anomalous behaviors

3.1. Anomalous effective thermal conductivity

From Eq. (28), we arrive at $\lambda_{eff} = \langle\lambda(t)\rangle$ for the time-independent temperature distribution. Then, one can find that Eq. (28) leads to $\lambda_{eff} \propto t^{-n}$. Furthermore, Eq. (24) is also the diffusivity distribution of $D_{th} = \lambda/c$, which will give rise to a non-Brownian MSD, namely, $\langle|\Delta\mathbf{x}|^2\rangle \propto t^{1-n}$. Thereby, the time-dependence of λ_{eff} can be connected to the temporal growth of the MSD ($\langle|\Delta\mathbf{x}|^2\rangle \propto t^{\alpha}$), $\lambda_{eff} \propto t^{\alpha-1}$. This connection has been observed in many previous investigations, i.e., the Lévy-walk (LW) model [39] and linear response theory [13]. Since λ_{eff} will diverge to infinity in superdiffusive heat conduction and converge to zero in subdiffusive heat conduction as $t \to +\infty$, the "transit time" $t_T = L/v$ is commonly introduced as a cut-off time, wherein v is the sound velocity. Thereby, it becomes a length-dependence, $\lambda_{eff} \sim L^{\alpha-1}$. The "transit time" is widely used in the Green-Kubo formula but not rigorous, which will be invalid in the absence of a sound wave, for instance, $t \propto L^{\mu}$ with $\mu = 1.5 \pm 0.001$ [40]. For anomalous heat diffusion paired with $\langle|\Delta\mathbf{x}|^2\rangle \propto t^{\alpha}$, we have $\rho_\lambda = t^{1-\alpha} f(\lambda t^{1-\alpha})$. Noting that in the renormalized phonons and effective phonon theory, ρ_λ is given by Eq. (11). The distribution for the contributions from different phonon modes, ρ_q, can be calculated through numerical methods for specific materials or models. Thus, one can verify the superstatistical approach in anomalous heat diffusion by comparing Eq. (11) and $\rho_\lambda = t^{1-\alpha} f(\lambda t^{1-\alpha})$ in numerical calculations.

3.2. Anomalous asymptotics

In this section, we focus on non-Fourier behaviors for time-dependent temperature distributions. We consider well-posed problems of Eq. (16), which have constant material properties and time-independent asymptotic states $T_a(\mathbf{x}) = \lim_{t\to+\infty} T(\mathbf{x}, t|\lambda)$. For Eq. (16), one common type of solutions takes the following form of separated variables

$$T(\mathbf{x}, t|\lambda) = T_a(\mathbf{x}) + \sum_{i=1}^{+\infty} \exp(-\lambda e_i t) A_i(\mathbf{x}) \tag{41}$$

In Eq. (41), $\{e_i\} \in \mathbb{R}^+$ denote the eigenvalues, while $\{A_i(\mathbf{x})\} \in \mathbb{R}$ denote the solutions of the spatial eigen-equations, which are determined by the initial conditions. For the exponential ρ_λ, the overall temperature distribution is given by

$$T_o(\mathbf{x}, t) = \int_0^{+\infty} T(\mathbf{x}, t|\lambda)\rho_\lambda(\lambda) d\lambda = T_a(\mathbf{x}) + \sum_{i=1}^{+\infty} \frac{A_i(\mathbf{x})}{\langle\lambda\rangle e_i t + 1}. \tag{42}$$

As comparison, we consider the solution of Fourier heat conduction with the same λ_{eff}, which is written as

$$T_F(\mathbf{x}, t) = T_a(\mathbf{x}) + \sum_{i=1}^{+\infty} \exp(-\langle\lambda\rangle e_i t) A_i(\mathbf{x}). \tag{43}$$

Eq. (42) exhibits a power-law decay to the asymptotic state for each e_i, $[\langle\lambda\rangle e_i t + 1]^{-1} \sim t^{-1}$, whereas Eq. (43) decays exponentially. Power-law ρ_λ is another simple case [27]:

$$\rho_\lambda(\lambda) = \frac{\lambda_0^\mu}{(\lambda_0 + \lambda)^{1+\mu}},\ \lambda_0 = \frac{\langle\lambda\rangle}{\mu \mathrm{B}(2, \mu - 1)}, \tag{44}$$

where $\mu > 0$ and $\mathrm{B}(2, \mu - 1)$ is the beta function. The corresponding $T_o(\mathbf{x},t)$ is

$$T_o(\mathbf{x}, t) = T_a(\mathbf{x}) + \sum_{i=1}^{+\infty} \mu(\lambda_0 e_i t)^\mu \exp(\lambda_0 e_i t)\Gamma(-\mu, \lambda_0 e_i t) A_i(\mathbf{x}), \tag{45}$$

where $\Gamma(-\mu, \lambda_0 e_i t) = \int_{\lambda_0 e_i t}^{+\infty} \varepsilon^{-\mu-1} \exp(-\varepsilon) d\varepsilon$ is the incomplete gamma function. The unsteady-state term in Eq. (45) still satisfies power-law decay as $t \to +\infty$. Obviously, the power-law decay is much slower than the exponential decay. Accordingly, despite the same λ_{eff}, the superstatistical model predicts slower long-time asymptotics than classical Fourier's law. We show this conclusion by two specific ρ_λ, while this conclusion holds for arbitrary ρ_λ which is piecewise continuous. As $t \to +\infty$, $T(\mathbf{x}, t|\lambda)$ with $\lambda \in [\langle\lambda\rangle, +\infty)$ will tend to $T_a(\mathbf{x})$ rapidly. Consequently, $T(\mathbf{x}, t|\lambda)$ with $\lambda \in [0, \langle\lambda\rangle]$ will play the predominant role in asymptotics. This feature implies that the superstatistical model will predict smaller effective thermal diffusivity D_{eff} than λ_{eff}/c in steady-state problems, which can be significant in experimental methods, i.e., the flash method [41]. In this method, material properties are constant, and the solution of Eq. (16) is written as

$$T(x, t|\lambda) = T|_{t=0} + \frac{Q}{cL} + 2\sum_{i=1}^{+\infty} \frac{Q}{cL} \cos\frac{i\pi x}{L} \exp\left(-\frac{\lambda i^2 \pi^2 t}{cL^2}\right), \tag{46}$$

where Q is the radiant energy and $T|_{t=0}$ is the initial temperature. For classical Fourier heat conduction, $\rho_\lambda = \delta(\lambda - \langle\lambda\rangle)$ and $\langle\lambda\rangle$ satisfies

$$\frac{\langle\lambda\rangle}{c} = 0.14\frac{L^2}{t_{0.5}}, \tag{47}$$

where $t_{0.5}$ is the time required for the back surface to reach half of the maximum temperature rise $\Delta T_{\max}$. For the superstatistical model with exponential ρ_λ, the overall temperature distribution reads

$$T_o(x, t) = T|_{t=0} + \frac{Q}{cL} + 2\sum_{i=1}^{+\infty} \frac{Q}{cL} \cos\frac{i\pi x}{L} \frac{cL^2}{\langle\lambda\rangle i^2\pi^2 t + cL^2}, \tag{48}$$

and $t_{0.5}$ is calculated as

$$\frac{\langle\lambda\rangle}{c} = 0.24\frac{L^2}{t_{0.5}}. \tag{49}$$

Due to the superstatistical effects, $t_{0.5}$ is larger than the classical result in Eq. (47). If one still uses Eq. (47) to calculated the thermal conductivity, the thermal conductivity will be underestimated. Hence, the superstatistical effects can be considered as a possible explanation for the deviation between unsteady-state and steady-state experimental results. Furthermore, one can define t_δ where $0 < \delta < 1$ and t_δ stands for the time required for the back surface to reach $\delta\Delta T_{\max}$. For each given δ, $\lambda_{eff} = \lambda_{eff}(\delta)$ can be calculated by comparing the experimental results with the solution in Eq. (46). In the superstatistical model, $\lambda_{eff}(\delta)$ must decrease with δ increasing because $T(\mathbf{x}, t|\lambda)$ with $\lambda \in [0, \langle\lambda\rangle]$ will play the predominant role in long-time

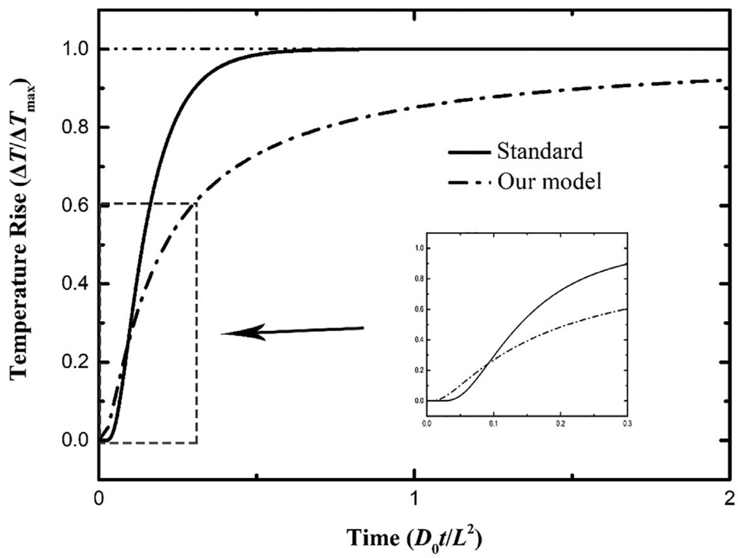

Fig. 2. Temperature rise in the flash method.

asymptotics. This is anomalous behavior because a temperature evolution corresponds to variable λ_{eff}, and if it is confirmed by measurement, one can use the superstatistical model to describe it. The corresponding difference between the standard case and our model is shown in Fig. 2 with $D_0 = \langle\lambda\rangle/c$. We can observe that the superstatistical model yields much slower long-time asymptotics than the standard case. This behavior also exists in corresponding numerical solutions. A recent numerical study [42] of poly-3,4-ethylenedioxythiophene (PEDOT) is an example. This study is also based on unsteady-state heat conduction and predicts $\lambda_{eff}\propto t^{\alpha-1}$ for anomalous heat diffusion with $\langle|\Delta\mathbf{x}|^2\rangle\propto t^{\alpha}$, which agrees with our result. If the superstatistical model holds, the long time numerical solution will exhibit the same anomalous trend mentioned above.

4. Discussion

Strictly speaking, the diffusing thermal conductivity model, is not exactly equivalent to the case with given ρ_λ. That is because this model involves two irreversible processes, heat conduction and advection-diffusion of the thermal conductivity. Both of the two processes are paired with positive entropy production rates. Thus, the total entropy production rate Φ includes not only heat conduction but also advection-diffusion of the thermal conductivity, which must be larger than the case with given ρ_λ. The simplest example is the 1D steady-state problem with a linear temperature distribution:

$$T(x) = T(x=0) + \frac{T(x=L) - T(x=0)}{L}x. \tag{50}$$

Then, the total entropy production rate Φ equals to the sum of the part by heat conduction

$$\Phi_h = Ak_B \int_0^{+\infty} \rho_\lambda(\lambda) d\lambda \int_0^L \frac{\lambda}{T^2(x)} \left|\frac{dT(x)}{dx}\right|^2 dx, \tag{51}$$

and the part by advection-diffusion of the thermal conductivity

$$\Phi_\lambda = Ak_B \int_0^L dx \int_0^{+\infty} \frac{1}{K_\lambda(\lambda)\rho_\lambda(\lambda)} |J_\lambda(\lambda)|^2 d\lambda, \tag{52}$$

with A denoting the cross sectional area.

The superstatistical heat conduction can also be extended in other non-Fourier models. For constant material properties, Fourier's law predicts instantaneous responses everywhere to a sudden thermal disturbance at an arbitrary point. It indicates infinite speeds of heat propagation [28], which is unphysical. "The most obvious and simple generalization of Fourier's law that will give rise to finite speeds of propagation" [28] is the following Cattaneo equation [43]

$$\mathbf{q}(\mathbf{x}, t) + \tau \frac{\partial \mathbf{q}(\mathbf{x}, t)}{\partial t} = -\lambda \nabla T(\mathbf{x}, t), \tag{53}$$

where τ is the relaxation time. The Cattaneo equation educes a hyperbolic governing equation

$$c\frac{\partial T(\mathbf{x}, t)}{\partial t} + c\tau \frac{\partial^2 \mathbf{q}(\mathbf{x}, t)}{\partial t^2} = \nabla \cdot [\lambda \nabla T(\mathbf{x}, t)], \tag{54}$$

which transmits heat wave with a finite speed $\sqrt{\lambda/c\tau}$. The thermal relaxation can be generalized into a superstatistical form by introducing a distribution of the relaxation time, ρ_τ. Then, Eq. (53) is rewritten as

$$\mathbf{q}(\mathbf{x},t|\tau,\lambda)+\tau\frac{\partial\mathbf{q}(\mathbf{x},t|\tau,\lambda)}{\partial t}=-\lambda\nabla T(\mathbf{x},t|\tau,\lambda), \tag{55}$$

while the overall temperature distribution and heat flux are given by

$$T_o(\mathbf{x},t)=\int_0^{+\infty}\int_0^{+\infty}T(\mathbf{x},\,t|\tau,\lambda)\rho_\tau\rho_\lambda d\lambda d\tau, \tag{56}$$

$$\mathbf{q}_o(\mathbf{x},t)=\int_0^{+\infty}\int_0^{+\infty}\mathbf{q}(\mathbf{x},\,t|\tau,\lambda)\rho_\tau\rho_\lambda d\lambda d\tau. \tag{57}$$

Such models can be applied to ballistic heat transport [44,45]. Fractional-order derivatives are widely used to describe anomalous diffusion [46], which are also applied to anomalous heat conduction [17–19]. In recent years, the fractional-order approach has been extended in forms similar to those of superstatistics, by introducing distributed-order fractional operators in the diffusion equation [47]. The corresponding order distribution fulfills the normalization condition similarly. In the spirit of this, we generalize the superstatistical approach to apply to the fractional-order models. The superstatistical generalization of the fractional-order heat conduction model can be proposed as

$$\mathbf{q}(\mathbf{x},t|\beta)=\tau_\beta^{1-\beta}\,{}_0^{RL}D_t^{1-\beta}[\lambda\nabla T(\mathbf{x},t|\beta)], \tag{58}$$

$$T_o(\mathbf{x},t)=\int_0^1 T(\mathbf{x},\,t|\beta)\rho_\beta d\beta, \tag{59}$$

$$\mathbf{q}_o(\mathbf{x},t)=\int_0^1 \mathbf{q}(\mathbf{x},\,t|\beta)\rho_\beta d\beta, \tag{60}$$

where ρ_β is the distribution of the order β.

5. Conclusions

The superstatistical framework is introduced into the modeling of heat conduction. The superstatistical model obeys Fourier's law and the continuity equation at the individual level. Based on the work by Chubynsky and Slater [26], we also propose a diffusing thermal conductivity model for the evolution of the thermal conductivity distribution, which satisfies the advection-diffusion equation. The connection between the superstatistical frameworks for the PDF and temperature distribution is also discussed based on CIT and BG statistical mechanics. We show that both approaches reflect a same entropy transport process.

The superstatistical model can predict anomalous behaviors in heat conduction. It can give rise to time-dependent effective thermal conductivity in a time-independent temperature distribution. In the diffusing thermal conductivity model, the time-dependence is power-law and can be connected the MSD, which coincides with existing investigations. For the time-dependent temperature distribution, the superstatistical model predicts slower long-time asymptotics than the classical Fourier's law, which may have effects on related experiments such as the flash method.

Different from the case with a given thermal conductivity distribution, the diffusing thermal conductivity model involves two irreversible processes, heat conduction and advection-diffusion of the thermal conductivity. The difference can be reflected by the total entropy production rate. The superstatistical framework can also be extended into other non-Fourier models including the Cattaneo and fractional-order heat conduction models.

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